PROBLEM SET 6

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In the following exercises X, Y is are locally compact Hausdorff spaces.

Problem 1. *Let μ, ν be Radon measures on X, Y, not necessarily σ-finite. If f is a nonnegative l.s.c. function on $X \times Y$, show that $x \to \int f_x d\nu$ and $y \to \int f^y d\mu$ are Borel measurable and $\int f d(\mu \hat{\times} \nu) = \iint f d\mu d\nu = \iint f d\nu d\mu$.*

Proof. The result follows verbatim from the argument of Proposition 7.25, once χ_U is replaced by f. □

Problem 2. *The following gives an example of a smooth function not equal to its Taylor expansion at 0. Let $f(t) = e^{-1/t}$ for $t > 0$, $f(t) = 0$ for $t \leq 0$. Check that*

(1) For $k \in \mathbb{N}$ and $t > 0$, $f^{(k)}(t) = P_k(1/t)e^{-1/t}$ where P_k is a polynomial.
(2) $f^{(k)}(0)$ exists and is equal to 0 for all k.

Proof. (1) Inductively one shows $P_{k+1}(x) = x^2(P_k(x) - P_k'(x))$.
(2) Inductively one shows

$$f^{k+1}(0) = \lim_{t \to 0^+} f^{(k)}(t)/t = \lim_{x \to +\infty} x P_k(x)/e^x = 0 = \lim_{t \to 0^-} f^{(k)}(t)/t.$$

□

Problem 3. *If $f \in L^\infty$ and $\|f^y - f\|_\infty \to 0$ as $y \to 0$, then f agrees a.e. with a uniformly continuous function.*

Proof. Consider $A_r f(x) = \frac{1}{m(B(r,x))} \int_{B(r,x)} f(y)dy$ for $r > 0$. Since $f \in L^\infty$, we have $\int_K |f| \leq \|f\|_\infty \cdot m(K) < \infty$ for any compact set K, so $f \in L^1_{loc}$. Therefore by Lemma 3.16, $A_r f$ is continuous and $\lim_{r \to 0} A_r f = f$ a.e.. Hence it suffices to show $\{A_r f\}_r$ is uniformly continuous and uniformly Cauchy. Notice that for $r > 0$ we have

$$|A_r f(x-y) - A_r f(x)| \leq \|f\|_\infty \cdot \int |\frac{\chi_{B(r,x-y)}}{m(B(r,x-y))} - \frac{\chi_{B(r,x)}}{m(B(r,x))}|,$$

it then follows $A_r f$ is uniformly continuous since Lebesgue measure is translation invariant. To see $\{A_r f\}_r$ is uniformly Cauchy, we use

$$|A_r f(x) - f(x)| \leq \frac{1}{m(B(r,x))} \int_{B(r,x)} |f(z) - f(x)| dz \leq \sup_{|y|<r} \|f^y - f\|_\infty,$$

and thus

$$\|A_r f - A_s f\|_\infty \leq \sup_{|y|<r} \|f^y - f\|_\infty + \sup_{|y|<s} \|f^y - f\|_\infty \to 0$$

as $r, s \to \infty$. □